Journal of Structural Geology, Vol. 2, No. 3, 1980, pp. 379 to 382.
© Pergamon Press Ltd. 1980. Printed in Great Britain
0191-8141/80/0901-0379 $02.00/0

Compatibility of two-dimensional strains and rotations along strain trajectories

P. R. COBBOLD

Centre Armoricain d'Etude Structurale des Socles (C.N.R.S.), Université de Rennes, Campus de Beaulieu, 35042 Rennes-Cédex, France

(Received 24 December 1979; accepted in revised form 3 March 1980)

Abstract—A study of the compatibility of strains and rotations is necessary for a proper understanding of deformation patterns. Principles of compatibility also underlie all methods for removing the effects of deformation and finding the pretectonic shape of deformed regions. The object of this paper is to derive compatibility equations in a direct way and in a simple form which can be interpreted geometrically. This is done using sets of orthogonal curvilinear coordinates parallel to strain trajectories. As a result, all shear components vanish and only principal strains appear in the equations. Such advantages are obtained at the expense of the extra complexity of curvilinear coordinates compared with Cartesian ones. Orthogonal curvilinear coordinates are described first, with particular reference to curvature. Compatibility of strains and rotations is then discussed by comparing curvatures of trajectories in the undeformed and deformed states. Results are compared with previous theoretical work and numerical methods.

INTRODUCTION

AS GEOLOGISTS have obtained more and more strain data from orogenic regions, they have become interested in methods of geometrically removing the strain so as to discover original pretectonic configurations and evaluate tectonic displacements. The problem is essentially a geometric one, but it can be complex if deformation is variable in space, as it generally is in nature. Some progress in finding practical methods of strain removal has been made by geologists (Oertel 1974, Cobbold 1977, Schwerdtner 1977, Oertel & Ernst 1978, Hossack 1978, Cobbold 1979), but the basic theory, as reviewed by Truesdell & Toupin (1960), has not been systematically explored with this object in mind.

The geologist's problem can be stated as follows. The deformation of an infinitesimal element of rock can be expressed in terms of three components: a rigid translation, a rigid rotation, and a strain or shape change. The first two of these do not modify the shape or internal structure of the element of rock: unless the geologist knows its original position and orientation, which he seldom does, he cannot determine components of translation and rotation in a direct way. In contrast, the strain causes changes in shape and internal structure which are usually recognizable and can often be measured. Thus the problem is to reverse a deformation given strain data only. Truesdell & Toupin (1960, p. 271) note that this problem is soluble provided the strains satisfy various conditions of compatibility. This means that certain components cannot vary in space independently of others.

As examples of the idea of compatibility, Cobbold (1977) and Schwerdtner (1977) have shown how variations in strain are necessarily accompanied by variations in rigid body rotation. This is now the basis for certain numerical methods of strain removal (Schwerdtner 1977, Cobbold 1979). Using Cartesian coordinates and working in two dimensions, Cobbold (1977) has derived expressions giving rotation gradients in terms of strain and strain gradients. Unfortunately these expressions are complex enough for their geometric meaning to be hidden and they are cumbersome to use. This complexity increases for three-dimensional deformations. Thus there is a need for simpler expressions whose meaning can be grasped. The object of this paper is to derive such expressions.

As hinted earlier (Cobbold 1979), considerable simplification can be obtained by using special systems of orthogonal curvilinear coordinates which are parallel at all points to the principal directions of the strain ellipsoid. In terms of these coordinates, (a) shear components of the deformation tensor vanish, and (b) normal components are proper numbers. Thus we only need to deal with principal strain values. These advantages are somewhat but not completely offset by the need to describe the curvature and local scale factors for the curvilinear coordinates. Orthogonal curvilinear coordinates are basically simple to use but are perhaps not familiar to some geologists. Hence this paper starts by describing some simple properties of these coordinates, including conditions of compatibility that will then be applied to strains and rotations. The analysis here is two-dimensional, but an extension to three dimensions is more simple than it is with Cartesian coordinates. As far as possible, the notation and terminology used here are those of Truesdell & Toupin (1960).

ORTHOGONAL CURVILINEAR COORDINATES

Orthogonal curvilinear coordinates were introduced and studied extensively by Lamé (1859). They found immediate applications, such as to the theory of elasticity (Love 1892). A very good introduction is provided by Borg (1963, Chapter 3). The original theory of Lamé is sufficient for the present discussion, but the more preva-

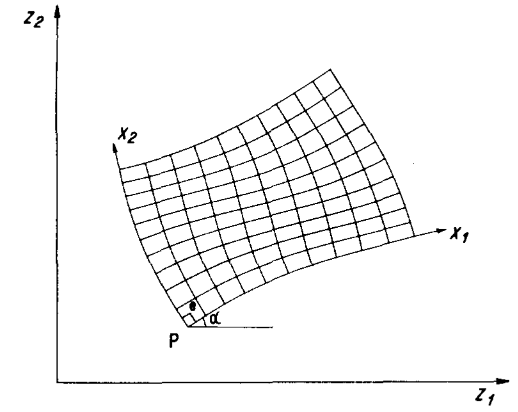

Fig. 1. Orthogonal curvilinear coordinates (x_1, x_2) referred to a Cartesian frame (z_1, z_2).

lent tensor notation, as used by Truesdell & Toupin (1960) will also be referred to in passing.

Consider a system of Cartesian coordinates (Fig. 1) and a point P (z_1, z_2). Suppose we choose two functions of z_1 and z_2 as follows:

$$\begin{aligned} x_1 &= f_1(z_1, z_2), \\ x_2 &= f_2(z_1, z_2). \end{aligned} \tag{1}$$

We assume that these functions and their inverse are single-valued and as many times differentiable as required. They will have constant values (for example, $x_1 = a$) along two families of curves which we may choose as curvilinear coordinates. Thus the position of P is also given by $P(x_1, x_2)$. Equation (1) is a coordinate transformation: it describes how z_1, z_2 transform to become x_1, x_2. Notice that this in no way implies a deformation of the material, but merely a change in the way of describing it.

From the inverse of (1), using the chain rule of differentiation, we obtain:

$$\begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}. \tag{2}$$

So far, the functions (1) have not been chosen in any special way that will guarantee the curvilinear coordinates to be orthogonal. There are several ways of explaining how the choice is made. Here we will use a non-rigorous but graphical argument. We require (2) to describe how a small non-Cartesian element, e (Fig. 1), is transformed into an equivalent Cartesian element.

Since both elements each have orthogonal sides, the transformation involves only two changes: first, a change in the lengths of the sides; second, a change in orientation. Thus the matrix in (2) is expressed as the product of two other matrices:

$$\begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}. \tag{3}$$

The first matrix (on the right) is a diagonal one. The quantities h_1 and h_2 are simply scale factors: they give the true (Cartesian) lengths of the sides of the element of the curvilinear coordinate system. As the interval between any two adjacent coordinate lines varies, so does h_1 and h_2. The reciprocals of h_1 and h_2 are called magnification factors by Borg (1963). The squares of h_1 and h_2, denoted g_{11} and g_{22} are elements of the metric tensor, g_{ij}. For oblique curvilinear coordinates, the metric tensor is the most convenient measure in terms of Cartesian coordinates. For the orthogonal coordinates used here, expressions are more simply written in terms of h_1 and h_2.

The second matrix [on the rhs of (3)] is an orthogonal matrix. It expresses a rotation of the coordinate lines through an angle α. If this angle is zero, (2) becomes equivalent to $dz_1 = h_1\, dx_1$ and $dz_2 = h_2\, dx_2$.

COMPATIBILITY AND CURVATURE

It is perhaps intuitively apparent that in order to maintain coordinates orthogonal at all points, the quantities h_1, h_2 and α cannot vary independently in space. They must be compatible. Compatibility equations can be derived in the following way.

In (2), the components of the matrix are not all independent; by definition, they must satisfy the following conditions:

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(\frac{\partial z_1}{\partial x_1}\right) &= \frac{\partial}{\partial x_1}\left(\frac{\partial z_1}{\partial x_2}\right), \\ \frac{\partial}{\partial x_2}\left(\frac{\partial z_2}{\partial x_1}\right) &= \frac{\partial}{\partial x_1}\left(\frac{\partial z_2}{\partial x_2}\right). \end{aligned} \tag{4}$$

If we multiply out the two matrices on the rhs of (3) and apply the conditions (4), we obtain, after a little ordering:

$$\begin{aligned} \frac{\partial \alpha}{\partial x_1} &= -\frac{1}{h_2}\frac{\partial h_1}{\partial x_2}, \\ \frac{\partial \alpha}{\partial x_2} &= \frac{1}{h_1}\frac{\partial h_2}{\partial x_1}. \end{aligned} \tag{5}$$

These first order partial differential equations are compatibility equations for α in terms of h_1 and h_2. If we write $ds_1 = h_1\, dx_1$ and $ds_2 = h_2\, dx_2$ for the arc lengths of an infinitesimal element (Fig. 2), then (5) becomes:

$$\begin{aligned} \frac{\partial \alpha}{\partial s_1} &= -\frac{1}{h_1}\frac{\partial h_1}{\partial s_2} = c_1 = \frac{1}{r_1}, \\ \frac{\partial \alpha}{\partial s_2} &= \frac{1}{h_2}\frac{\partial h_2}{\partial s_1} = c_2 = \frac{1}{r_2}, \end{aligned} \tag{6}$$

where c and r denote curvature and radius of curvature, respectively. The quantity $\partial\alpha/\partial s_1$ is the curvature of the x_1-lines by definition. The quantity $\partial h_1/\partial s_2$ is the variation in spacing of x_1-lines with true distance along x_2. Divided by the reference length h_1, this also gives the curvature. The expressions are analogous to those used in the theory of simple bending, but notice that in (6) it is the coordinate frame that is bent, not the material. Equation (6a) gives the curvature of the x_1-lines, (6b) that of the x_2-lines.

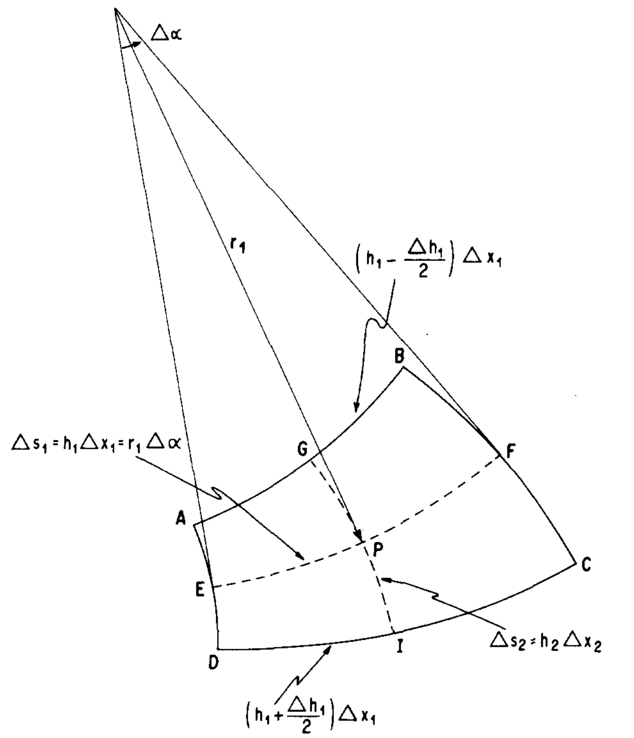

Fig. 2. Element of curvilinear grid, showing relationship between curvature and grid spacing. Arc-length of curve EF is Δs_1. Increase in arc-length from AB to DC is $\Delta h_1 \, \Delta x_1$. Divided by Δs_2 this gives a gradient $\Delta h_1 \, \Delta x_1/\Delta s_2$. Divided again by the reference length $\Delta s_1 = h_1 \, \Delta x_1$, this gives $\Delta h_1/h_1 \, \Delta s_2$. In the limit as the element becomes small, this is the curvature, $(1/h_1)\partial h_1/\partial s_2 = 1/r_1$ (see equation 6).

In (5) the terms in α can be eliminated by differentiating the first equation with respect to x_2, the second with respect to x_1 and equating. This yields:

$$\frac{\partial}{\partial x_2}\left(\frac{1}{h_2}\frac{\partial h_1}{\partial x_2}\right) + \frac{\partial}{\partial x_1}\left(\frac{1}{h_1}\frac{\partial h_2}{\partial x_1}\right) = 0. \qquad (8)$$

This second order partial differential equation expresses the compatibility of h_1 and h_2. Any set of h_1, h_2 must satisfy (8) if the coordinates are to be orthogonal. One geometrical interpretation is that the curvature of the two sets of coordinate curves cannot vary independently if they are to remain orthogonal.

COMPATIBILITY EQUATIONS FOR DEFORMATION

The results of the last section will now be used to discuss compatibility of rigid rotations and finite strains along strain trajectories.

Consider two sets of orthogonal curvilinear coordinates, one parallel to strain trajectories in the deformed state, the other parallel to trajectories in the undeformed state (Fig. 3). Both sets are referred to a common Cartesian frame. Quantities in the deformed state will be described with minuscules, those in the undeformed state with majuscules. Thus equations (1)–(8) apply to the deformed state. If we write the same expressions with capital letters, then these will be applicable to the undeformed state. Thus the equivalent of (5) is:

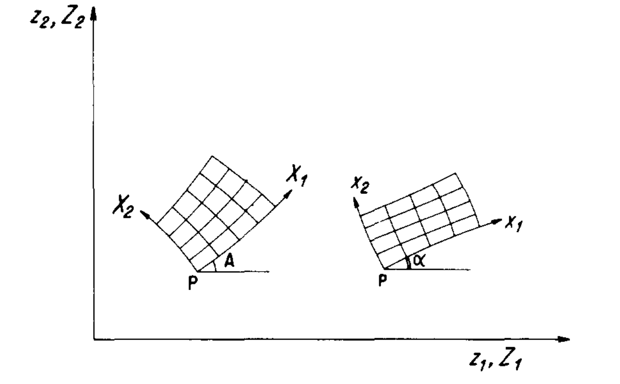

Fig. 3. Common Cartesian frame and sets of orthogonal curvilinear coordinates parallel to strain trajectories in the undeformed state (X_1, X_2) and deformed state (x_1, x_2).

$$\frac{\partial \mathrm{A}}{\partial X_1} = -\frac{1}{H_2}\frac{\partial H_1}{\partial X_2}, \qquad \frac{\partial \mathrm{A}}{\partial X_2} = \frac{1}{H_1}\frac{\partial H_2}{\partial X_1}. \qquad (9)$$

It has been argued before (Cobbold 1979) that the deformation transforms trajectories in the undeformed state into trajectories in the deformed state. The trajectories undergo changes in length (strain) and orientation (rigid rotation). At any point, the new length divided by the original length of a segment of trajectory is just the principal stretch in that direction, as assumed by Hossack (1978). Although it is not strictly necessary to do so, we will assume for convenience that the coordinate frame distorts with the material. This has two consequences. First, it means that x and X can be interchanged freely in all expressions. Second, the coordinate intervals H_1 and H_2 stretch to become the new intervals, h_1 and h_2, as follows:

$$h_1 = \lambda_1 H_1; \quad h_2 = \lambda_2 H_2. \qquad (10)$$

Here λ is the stretch (final length/original length) as defined by Truesdell & Toupin (1960, p. 255). The rigid body rotation θ at any point is equal to the difference in orientation between corresponding trajectories in the undeformed and deformed states:

$$\theta = \mathrm{A} - \alpha. \qquad (11)$$

If this quantity varies in space (as it must if deformation is heterogeneous), then the curvature of the trajectories either increases or decreases on passing from one state to the other. Thus the trajectories undergo changes in length and curvature. Any kind of continuous deformation can be expressed in terms of these changes.

From these considerations, it follows that gradients of rigid body rotation (giving changes of curvature) are related to gradients of stretch. This can be seen by substituting (10) into (9) and then using (11) and (5):

$$\frac{\partial \theta}{\partial x_1} = -\frac{\lambda_2}{h_2}\frac{\partial(h_1/\lambda_1)}{\partial x_2} + \frac{1}{h_2}\frac{\partial h_1}{\partial x_2}, \qquad \frac{\partial \theta}{\partial x_2} = \frac{\lambda_1}{h_1}\frac{\partial(h_2/\lambda_2)}{\partial x_1} - \frac{1}{h_1}\frac{\partial h_2}{\partial x_1}. \qquad (12)$$

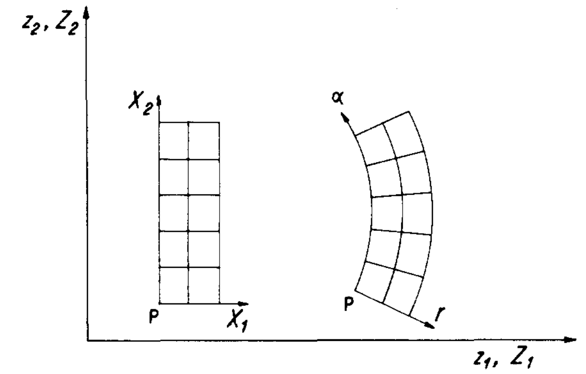

Fig. 4. Simple bending of a beam. Coordinates in undeformed state (left) are straight lines; those in deformed state (right) are plane polar.

This completes the analysis of compatibility in two dimensions.

AN EXAMPLE: SIMPLE BENDING OF A BEAM OR FOLD

The simplest example of a heterogeneous plane deformation is probably the bending of a beam or layer of material (Fig. 4). In the deformed state, the principal stretches are everywhere either parallel or normal to the layer boundaries. The strain trajectories are arcs and radii of circles, which we may conveniently describe by means of plane polar coordinates, r and α. If the origin of these is chosen to coincide with the origin of the common Cartesian frame, then the inverse of (1) becomes:

$$\begin{aligned} z_1 &= r\cos\alpha, \\ z_2 &= r\sin\alpha. \end{aligned} \tag{13}$$

Hence (2) becomes:

$$\begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -r\sin\alpha \\ \sin\alpha & r\cos\alpha \end{pmatrix} \begin{pmatrix} dr \\ d\alpha \end{pmatrix}. \tag{14}$$

By comparison with (3), we see that $h_1 = 1$ and $h_2 = r$. If we also choose $r = 1$ at the neutral surface of the beam (where by definition $\lambda_1 = \lambda_2 = 1$), then in general, the amount of stretch increases with the radius of curvature, that is, $\lambda_2 = r$. From (10) we obtain $H_2 = 1$; from (9), A = constant (which we may conveniently take as zero).

This shows that in the undeformed state the strain trajectories are straight lines, as is perhaps obvious intuitively. From (12), we obtain $\theta = -\alpha$, which means that the amount of rigid rotation equals the attitude of the layering in the deformed state, as is also obvious.

DISCUSSION

Equations (5), (9), (10) and (11) are more simple and easier to interpret than corresponding equations in Cartesian coordinates derived by Cobbold (1977). If true values of strain are known at all points in a deformed region, then either the non-Cartesian or the Cartesian equations can be used to calculate rigid rotations and so remove the effects of deformation, either locally or regionally.

The non-Cartesian equations are probably best used as follows. From a pattern of strain trajectories, values of h_1 and h_2 are calculated at each point. These are multiplied (10) by the principal stretches to give H_1 and H_2. Graphical or numerical integration of (9) then gives the orientation of trajectories in the undeformed state. These can be constructed from the fields of H_1, H_2 and A. Shape changes can then be mapped from the deformed to the undeformed state or vice versa, using the trajectories as mapping coordinates (Cobbold 1979).

Analytical procedures such as these suffer from a major drawback: the solution is uniquely obtainable only if the strain values are mutually compatible. For example, H_1 and H_2 must satisfy (8). There is no guarantee that they will do so if the strain data are subject to error, as generally they are in geological examples. Therefore what is required is a strain distribution that satisfies compatibility conditions and also is a best-fit to the original strain data. So far, no practical analytical method has been found for doing this.

The geologist faces another problem: although he may have enough data to construct reasonably accurate trajectories, values of principal stretch are usually obtainable at only a limited number of points spaced at finite intervals. This suggests the use of numerical procedures based on finite differences or finite elements. Cobbold (1979) has suggested one such procedure in which any incompatibility between strain data can be overcome by allowing gaps and overlaps between adjacent finite elements. Another possibility is to replace (9) and (8) by their finite difference analogues: this is being explored at the moment.

Acknowledgements—I am grateful to A. J. Watkinson and J. P. Brun for constructive criticism of the manuscript.

REFERENCES

Borg, S. 1963. *Matrix-tensor Methods in Continuum Mechanics*. Van Nostrand, New York.

Cobbold, P. R. 1977. Compatibility equations and the integration of finite strains in two dimensions. *Tectonophysics* **39**, T1–T6.

Cobbold, P. R. 1979. Removal of finite deformation using strain trajectories. *J. Struct. Geol.* **1**, 67–72.

Hossack, J. R. 1978. The correction of stratigraphic sections for tectonic finite strain in the Bygdin area, Norway. *J. geol. Soc. Lond.* **135**, 229–241.

Lamé, G. 1859. *Leçons sur les Coordonées Curvilignes et Leurs Diverses Applications*. Mallet-Bachelier, Paris.

Love, A. E. H. 1892. *A Treatise on the Mathematical Theory of Elasticity*. Cambridge University Press, London.

Oertel, G. 1974. Unfolding of an antiform by reversal of observed strains. *Bull. geol. Soc. Am.* **85**, 445–450.

Oertel, G. & Ernst, W. G. 1978. Strain and rotation in a multilayered fold. *Tectonophysics* **48**, 77–106.

Schwerdtner, W. M. 1977. Geometric interpretation of regional strain analyses. *Tectonophysics* **39**, 515–531.

Truesdell, G. & Toupin, R. 1960. The classical field theories. *Handb. Phys.* **3**, 226–793.